Cambridge Assessment Admissions Testing

Sixth Term Examination Paper [STEP]

Mathematics 2 [9470]

2019

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2019

Examiner's Report

Introduction

The Pure questions were again the most popular of the paper, with only one of the questions attempted by fewer than half of the candidates (of the remaining four questions, only question 9 was attempted by more than a quarter of the candidates). In many of the questions candidates were often unable to make good use of the results shown in the earlier parts of the question in order to solve the more complex later parts. Nevertheless, some good solutions were seen to all of the questions. For many of the questions, solutions were seen in which the results were reached, but without sufficient justification of some of the steps.

Question 1

This was the question answered by the largest proportion of candidates and many good solutions were seen. However, many candidates did not appreciate the importance of the phrase *if and only if* in parts of this question. As a result a large number of attempts failed to achieve full marks as it was not made clear that the reasoning presented also worked in the opposite direction.

Having shown the first result, many candidates were able to identify the appropriate choice of $g(x)$ when attempting part (i) and successfully showed that $2a = q + r$. Many were also able to find a correct expression for the gradient, although some did not find this expression in terms of the variables requested.

In part (ii) a pleasing number of candidates were able to recognise that the results from part (i) would be relevant here as well. Again, some of the solutions to this part failed to recognise that the question required the result to be shown in both directions.

Question 2

This question was another popular question that was generally well answered, achieving the second-best average mark of all of the questions and was also the question for which the largest number of solutions received full marks. Most candidates drew a convincing sketch to demonstrate that the two integrals make a rectangle. Arguments from sketches showing the inverse function and reflective symmetry were less successful and often candidates' diagrams assumed x to be a fixed point of $f(t)$.

By far the most common mistake in the first part was to notice the solution $g(2) = 1$ but not to factorise and use the quadratic discriminant to show that no other solutions were possible. The conceptually difficult part was to use $g^{-1}(y) = y^3 + y$, and many candidates stopped just before this point.

In the final part, many candidates tried to apply the stem identity in its original form, without noticing that $h(0) \neq 0$. This was the most difficult part, and those who modified it correctly generally did well. Candidates sometimes failed to check that $h'(t) > 0$, but this was not necessary for those who used $h(t) = g(t + 2)$.

Question 3

While this was a popular question it was also the one where the average mark achieved by candidates was the lowest. In this question many of the results to be reached were given in the question. Students therefore need to recognise that it is necessary for solutions to be presented very clearly, and it is for this reason that many solutions in the first parts did not achieve full marks. For example, justifications of the generalised result for a set of n real numbers expressed in the form of an inductive proof were the most successful.

For most candidates the majority of marks were scored in the sections up to and including part (i)(b). Many candidates were then unable to see how to work in the cases where $|x| \geq 1$ for part (i)(c). In the final part, candidates were often unable to put the equation into the form that had been used in the earlier parts of the questions and therefore did not manage to reduce the possible values of the integer roots to a sufficiently small set.

Question 4

This was a well-answered question, but also one in which a fairly large number of solutions scored very low marks. The majority of candidates were able to evaluate the first product using the identify provided and most were then able to apply the same technique to simplify the first expression in part (i). Many students then differentiated, but some then struggled to manage the notation correctly to reach the second result requested in part (ii).

Part (iii) required some care to ensure that the sums and products were over the correct range, but those who managed to adjust correctly for this were then able to reach the required results.

Question 5

It was difficult to get full marks on this question, with most candidates struggling to correctly prove 'if and only if' statements in both directions.

Mostly, the two constant sequences were successfully found and then correctly rejected for sequences of period 2, but few thought to check that the other two solutions to the quartic did not also coincide with the constant sequences. Most candidates were able to use the discriminant to produce bounds on p, but many could not justify the strictness of the inequality, which was best done by considering the boundary cases separately.

The first request of the second part was answered well, with most using only the fact that it was a positive quadratic and a minority delving into the details of $f(x)$. Most candidates who reached this part of the questions correctly used the result $f(x) > x$ to show that $f\big(f(x)\big) = x$ has no solutions, but many overlooked the connection between the final part and part (i).

Question 6

Of the Pure questions, this was the question that had the lowest average mark, mainly due to the large number of attempts that did not manage to score any marks.

Many candidates seemed uncomfortable with this question which asked them to look at what information can be gleaned about differential equations without directly solving them. Many candidates decided that the only way to proceed was to solve the differential equation, and almost invariably this led to long and convoluted methods. Candidates seemed to have very little idea that the differential equation can be interpreted as the gradient of a curve at different points – it was simply an object on which certain methods had to be applied. A surprisingly small number of candidates realised that setting $\frac{dy}{dx} = 0$ could (and should) be done directly in the differential equation to find the locus of stationary points.

This was also a question which required candidates to bring a lot of disparate information together in the final sketches. A large number of candidates said things like the gradient was negative between two lines, but their sketch showed something different. Some who said that there should be stationary points on the line $y = x - 1$ and $y = x - 3$ drew their curve tangentially to these two lines instead.

Overall this was a question which really benefitted candidates who took a moment to stop and think about what was being suggested, rather than blindly applying methods.

Question 7

This was the least popular of the Pure questions. Good solutions to this question often included clear diagrams to enable the angles being discussed to be identified easily. Many of the candidates were able to calculate the value of $\boldsymbol{a} \cdot \boldsymbol{b}$ correctly, but often did not fully justify that the triangle ABC was equilateral.

For the second part, many candidates were again able to establish the relationship between scalar products, but less success was seen in identifying the type of quadrilateral. In the final part there were a large number of different approaches taken and many of these were completed successfully by some of the candidates.

Question 8

Many good solutions were seen to this question, but solutions often lacked clear enough justification to be awarded full marks. However, there were also a surprising number of candidates who did not manage to invert the 2x2 matrices successfully. Candidates who claimed that the function f was the determinant of the matrix were not able to score high marks as the solutions did not then demonstrate that the results were true of *any* function satisfying the property given.

The first two parts of this question were largely done well. The third part was found more difficult, with few candidates realising that $\begin{pmatrix} a & b \\ ka & kb \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} a & b \\ a & b \end{pmatrix}$. Those who did were then often able to provide a full solution, although often these were not fully justified. Several candidates instead used $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} a & b \\ k^{-1}a & k^{-1}b \end{pmatrix} = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$ to produce a solution which covered all cases apart from the one where $k = 0$. In some cases, candidates did not appear to consider $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ to be an example of a matrix in which the second row was a multiple of the first.

In part (iv) many candidates made use of the fact that $f(P) \neq 0$ without showing that this must be the case.

Question 9

This was the most popular of the Mechanics and Statistics questions, but also one of the questions that attracted a large number of solutions that received no marks.

Students seemed relatively good at setting up the kinematics equations in this question and most had the useful idea of differentiating. Somewhat fewer thought about using either completing the square or the quadratic discriminant to decide where the derivative was positive. The logic of the question was very poorly understood, with many students seeing the given inequality as the end point rather than the starting point of the question.

In the second part of part (i) it was important that students demonstrated not just that a time existed where the distance is decreasing, but that this time was in the acceptable domain of the question.

Part (ii) was conceptually very similar to part (i) but most students found the increased algebraic demand too much.

Question 10

As with so many questions, the big stumbling block for students was drawing a good diagram from the information, including all the relevant forces.

With “show that” questions it is beholden on candidates to explain their working. Equations which just appear and lead to the correct answer are not sufficient. In mechanics, it would be very helpful for students to say, for example, “Taking moments about point A for the rod” or “Resolving for the string -rod system vertically” to give some sense of where an equation arises.

The flow of logic is a fundamental idea in mathematics, but it was clear in this question that it was not familiar to the vast majority of students. The questions effectively asked “if <given condition> show that <mechanical outcome>”. Most students reversed this to show that “if <mechanical outcome> then <given condition>”. In this question, most arguments were reversible, but it still demonstrated a fundamental misunderstanding of what was being asked.

The other issue which flummoxed students was dealing with inequalities. There are different rules of algebra associated with inequalities and this is something which is frequently tested in STEP. Candidates would benefit from thinking carefully about things like when can one inequality be substituted into another, or when can an inequality be squared. The intuition from equalities was too often applied without thinking.

Question 11

Candidates got the correct number of pairs in the special cases $n_3 = 9$ and $n_3 = 10$ but sometimes the working was very unclear. A large majority found the expressions for general n, the most common error being a shift $n \to n + 1$ in the answer.

Those who could obtain the result given for odd n in part (ii) were generally able to find the corresponding result for even n too. A common error was to double count the number of pairs of rods and not to double the number of pairs which made a triangle. Many candidates failed to explain why the conditions of part (i) were relevant for forming triangles.

The most successful candidates in part (iii) counted the number of triples which make a triangle using a sum, and divided by $\binom{2M+1}{3}$, while those who conditioned on the largest rod and used conditional probability did less well. A common conceptual error was to assume that each integer was equally likely to appear as the largest rod, and candidates making this assumption lost many marks. Otherwise, algebraic errors were the most common. Candidates should remember that when an answer is given in the question, they need to take care to fully justify their answers.

Question 12

Almost all candidates who attempted this question were able to achieve full marks on the first part. In the second part, the values of the interquartile range and 2σ were generally found correctly, but then many candidates did not realise that squaring would eliminate the square roots from the values to be compared.

In the final part of the question some candidates failed to recognise that the $(k+1)^{th}$ term of the expansion was the term in x^k and gave the term in x^{k+1} instead. A good number were successful in finding the lower quartile and the median, but only a minority realised that $\mu^{-n} = \left(1 + \frac{1}{n}\right)^n$. Those that did were more successful in proving that $\mu > \left(\frac{1}{4}\right)^n$ than $\mu < \left(\frac{1}{2}\right)^n$.

STEP MATHEMATICS 2

2019

Hints and Solutions

Question 1

For the introductory part, first find an equation of the tangent to the curve at the point with $x = a$. An expression can then be found for the y-coordinate of the point on the tangent where $x = p$ and this can easily be shown to be equal to 0 if and only if $g'(a) = 0$.

In part (i), the first result follows by identifying that $g(x) = A(x - q)(x - r)$ allows the first result to be applied. The gradient of the tangent can be found by differentiating $f(x)$ and then the fact that $2a = q + r$ can be used to eliminate a from this expression.

In part (ii) the tangent at the point where $x = c$ is essentially another case of the tangent considered in part (i), so the gradient of this tangent can be deduced easily. By equating the gradients of the two tangents it can be deduced that $q - p = r - q$ (although care needs to be taken to justify the choice of square roots). The equation of the tangent at $x = q$ can also be found and so any other points of intersection between this tangent and the curve can be found. The result then follows easily.

Question 2

From a sketch of the function, it can be seen that the first integral corresponds to an area below the curve and the second integral corresponds to an area to the left of the curve. These two areas make a rectangle, whose area is clearly expressed by the expression on the right of the equation.

In the first part it is clear that the value of $g(0)$ satisfies $\left(g(0)\right)^3 + g(0) = 0$. Clearly $g(0) = 0$ satisfies this, but it is necessary to factorise and then show that the quadratic factor has no other real solutions. The second result can be seen by differentiating and observing that $(3g(t)^2 + 1)$ must be greater than 0.

To evaluate the integral, observe that $g^{-1}(s) = s^3 + s$ and apply the result shown at the start of the question.

For the second part it must be noted that this function does not satisfy the conditions for the initial result to be applied. However, it can be seen that $h(t) = g(t + 2)$.

It therefore follows that $h'(t) > 0$ and the values of $h(0)$ and $h(8)$ can be deduced. By considering a sketch of this function it can be seen how to modify the initial result to apply in this case.

Question 3

The initial result can be shown by considering the four possible combinations of signs for x_1 and x_2. Induction can then be used to prove the more general result.

In part (i)(a) the initial result can then be applied to show that the value of $f(x) - 1$ must be less than or equal to a polynomial in $|x|$. Furthermore, the coefficients must also be less than or equal to A and so the value must be less than or equal to a sum that can be seen to be a geometric sequence.

In part (i)(b) the previous result can be applied in the case $x = \omega$ and, since $|\omega| < 1$ it must be the case that $1 - |\omega| > 0$. Therefore, the inequality can be multiplied by $(1 - |\omega|)$ without changing the direction of the inequality. The required inequality then follows easily.

To show that the inequalities continue to hold if $|\omega| > 1$, observe that $\frac{1}{\omega}$ is a root of the polynomial $g(x) = 1 + a_{n-1}x + \cdots + a_1x^{n-1} + x^n$, as $g\left(\frac{1}{\omega}\right) = \frac{1}{\omega^n}f(\omega) = 0$. Since $g(x)$ has the same properties as $f(x)$, $\frac{1}{|\omega|}$ must also satisfy (*). It then only remains to consider the case $|\omega| = 1$.

For the final part, observe that division by 135 produces a polynomial that satisfies the conditions specified and so the bounds on the value of ω reduces the cases to be considered to $\omega = \pm 1$ and $\omega = \pm 2$.

Question 4

In the first part, if the expression to be evaluated is multiplied by $\sin\left(\frac{\pi}{9}\right)$, then three applications of the given identity can be used. A similar process can then be used to simplify the first expression in part (ii). For the sum, note that $\tan x$ is the derivative of $-\ln(\cos x)$ and so, the result can be obtained by taking logs of the first result and then differentiating term by term.

For the final part, first change to a finite product (from $k = 1$ to $k = n$) and then take the limit as $n \to \infty$. Note however, that the product in part (ii) started at $k = 0$, so the result in part (ii) needs to be modified before it can be applied.

In the same way, the sum can be modified to start at $k = 0$ (where $k = j - 2$) and then the result of part (ii) can be applied with $x = \frac{\pi}{4}$.

Question 5

In part (i) the values of a for which the sequence is constant can be found by solving the equation $a = f(a)$. The sequence will have period 2 if the equation $a = f(f(a))$ has a solution that is different from those for which the sequence is constant. Although the equation $a = f(f(a))$ is a quartic, it is clear that the values of a for which the sequence is constant will be solutions of this equation as well. This means that two factors of the quartic are known and so the remaining factor will be a quadratic. When considering the roots of this quadratic it must also be checked to confirm that the roots are not repeats of the values that give a constant sequence.

In part (ii), note that there cannot be a solution to the equation $f(a) = a$ and so it must be the case that either $f(x) > x$ for all x or $f(x) < x$ for all x (since f is a continuous function). It is clear that $f(x) > x$ for large values of x.

Since it must be that case that $f(x) > x$ for all x if the sequence is not constant, it must also be the case that $f\big(f(x)\big) > x$ for all x.

Finally, it can be seen that, in the case where $q = p$ the sequence is of the form in part (i) and so it should be possible to deduce a case in which there is no value of a for which the sequence has period 2, but there is a value of a for which the sequence is constant.

Question 6

In the first part, substituting $y = mx + c$ into the differential equation will allow the values of m and c to be deduced. Since stationary points must satisfy $\frac{dy}{dx} = 0$, substituting this into the differential equation shows that stationary points must lie on the given line. It then follows that solution curves with $k < 2$ cannot have stationary points as they would have to cross the straight-line solution that has already been found.

Given that the relationship between x and y for any stationary point is known, it is possible to differentiate the differential equation and evaluate $\frac{d^2y}{dx^2}$ for any stationary point.

Once the substitution provided has been applied, the new differential equation can be solved by separating the variables and have equations that can be sketched easily.

In the second part, the same approach as part (i) can be used to find the possible sets of values for m and c. The RHS of the differential equation can be considered a function of $y - x$ and this allows it to be factorised. Solving $\frac{dy}{dx} = 0$ then shows that x and y must satisfy one of two linear equations and the sign of $\frac{dy}{dx}$ can be deduced for points between these two lines.

The graph can then be sketched, remembering that the curve cannot cross the two straight-line solutions.

Question 7

In part (i), taking the scalar product of $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ with each of the vectors in turn produces a set of three equations from which it can be deduced that $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{c} \cdot \boldsymbol{a}$ and that any pair of them add up to -1. Alternatively, it can be observed that $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) = (-\boldsymbol{c}) \cdot (-\boldsymbol{c})$.

It can then be shown that the angle between any pair of these vectors is $120°$ and so a sketch shows that the triangle must be equilateral.

In part (ii), a similar approach will lead to the given result. Alternatively, the result can be obtained by observing that $(\boldsymbol{a_1} + \boldsymbol{a_2}) \cdot (\boldsymbol{a_1} + \boldsymbol{a_2}) = (-\boldsymbol{a_3} - \boldsymbol{a_4}) \cdot (-\boldsymbol{a_3} - \boldsymbol{a_4})$. For part (a), note that it must be the case that the angle between any pair of vectors is equal to the angle between the other two vectors.

For part (b) use the vector $(\boldsymbol{a_1} - \boldsymbol{a_2})$ to find the length of one side of the tetrahedron. From the fact that the tetrahedron is regular it can be deduced that $\boldsymbol{a_1} \cdot \boldsymbol{a_2} = \boldsymbol{a_1} \cdot \boldsymbol{a_3} = \boldsymbol{a_1} \cdot \boldsymbol{a_4}$. The side length can then be calculated.

Question 8

In part (i), the property of f means that $f(\boldsymbol{M}) = f(\boldsymbol{MI}) = f(\boldsymbol{M})f(\boldsymbol{I})$. Note that justification of $f(\boldsymbol{I}) = 1$ requires that $f(\boldsymbol{M}) \neq 0$.

In part (ii), note that $\boldsymbol{J}^2 = \boldsymbol{I}$ and so the value of $f(\boldsymbol{J})$ must be either 1 or -1. The second result of this part follows from application of the property of function f.

For part (iii), first show that $f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = 0$ by applying the result of part (ii) and then pre-multiply this matrix by $\boldsymbol{K}$ to obtain one in which the second row is a multiple of the first.

For part (iv), note that $\boldsymbol{P}^2 = \boldsymbol{K}^{-1}\boldsymbol{PK}$ in the case where $k = 2$. This leads to the fact that $f(\boldsymbol{P})$ must be either 0 or 1. The fact that $\boldsymbol{P}^{-1}$ exists can then be used to show that $f(\boldsymbol{P})$ cannot be 0.

Question 9

In part (i), the position vector of the particle at time t can be calculated. The distance OP is then the modulus of this vector. It is easier to differentiate the square of the distance with respect to time (which is sufficient as this will be increasing if and only if the distance is increasing). The resulting expression can be shown to be positive if $\sin\alpha < \frac{2\sqrt{2}}{3}$. Similarly, in the case where $\sin\alpha > \frac{2\sqrt{2}}{3}$ it is possible to identify a value of t for which the distance is certainly decreasing and show that this is before the moment at which the particle lands.

In part (ii), the vector QP can again be calculated and then the distance PQ found by taking the modulus. As in part (i) it is simpler to deal with PQ^2 rather than PQ. In this case, care must be taken with the inequality to check that both sides are positive before they are squared and used to justify that the distance is increasing throughout the flight of P.

Question 10

A diagram is very useful in this question. First, note that the triangle ABC must be isosceles and then take moments about A. In the case given in part (i) this then shows that $T > W$ and so the string will break.

In part (ii), resolve the forces vertically to find an expression for the reaction force and then this can be used to find an expression for the maximum possible value for the frictional force. W can then be eliminated using the equation in part (i) found by taking moments about A. Rearranging then leads to an expression that can be used to explain the required result.

For the third part, the values of k for which breaking and slipping occur can be found from the answers to part (i). These two values can be used to set up an inequality that must be satisfied in order for slipping to occur before the string breaks.

Question 11

In part (i), the numbers of ways of choosing the pairs can be found by checking the numbers of possible values for n_2 for each choice of n_1. A clear list of the possibilities for each case should then make generalised formulae for the cases $n_3 = 2n + 1$ and $n_3 = 2n$.

In part (ii), the possible combinations which lead to a triangle match those found in the first part of the question. There are $\binom{N-1}{2}$ possibilities for the shorter two rods if the length of the longest rod is known, so combining this with the answers to part (i) the probability can be calculated for each of the two cases to be considered.

In part (iii), the probability can be calculated by multiplying the probability in part (ii) for each possible length of the longest rod by the probability that that length is the longest of the three rods. Adding all of these together will result in the overall probability that the rods can form a triangle.

Question 12

Part (i) requires a simple integration to calculate the values of $E(X)$ and $E(X^2)$. The required result then follows algebraically.

In part (ii), use integration to find the values of the quartiles and hence the interquartile range. Square the two values to allow them to be compared with each other.

In part (iii), the binomial expansion should be easy to write down, but note that the $(k+1)^{th}$ term is the term in x^k, not x^{k+1}. The lower quartile and median can be evaluated by integration of $f(x)$. To show the inequalities, note that $\left(\frac{1}{\mu}\right)^n = \left(1+\frac{1}{n}\right)^n$ and that each term in the expansion is positive, so the value must be greater than the sum of the first two terms. Similarly, the $(k+1)^{th}$ term of the expansion can be shown to be greater than $\frac{1}{k!}$, so the result that may be assumed will lead to the other inequality.

STEP MATHEMATICS 2

2019

Mark Scheme

STEP II 2019 Mark Scheme

1			$f'(x) = g(x) + (x-p)g'(x)$ Tangent passes through $(a, (a-p)g(a))$ Equation of tangent is $y = \big(g(a) + (a-p)g'(a)\big)(x-a) + (a-p)g(a)$ (or equivalent equation) Substitution of $x = p$ into equation of tangent $y = -(a-p)^2 g'(a)$ Verification that if $g'(a) = 0$, then $y = 0$ If $y = 0$ then $g'(a) = 0$ because $a \neq p$	M1 M1 A1 E1 E1 E1 (AG) (6 marks)
	(i)		$g(x) = A(x-q)(x-r)$ identified $g'(a) = 0 \Rightarrow 2a = r + q$ (legitimately obtained) Gradient of tangent is $g(a) + (a-p)g'(a)$ $= A(a-q)(a-r)$ $= -\frac{1}{4}A(r-q)^2$	M1 A1 (AG) M1 A1 (4 marks)
	(ii)		By symmetry, the gradient of the second tangent is $-\frac{1}{4}A(p-q)^2$ (can be implied) Parallel iff $(p-q)^2 = (q-r)^2$ $\Leftrightarrow q - p = r - q$ since $p < q < r$. Tangent at $x = q$, $y = A(q-p)(q-r)(x-q)$, Meets curve again when $(q-p)(q-r)(x-q) = (x-p)(x-r)(x-q)$ $\Leftrightarrow (q-p)(q-r) = (x-p)(x-r)$ since $x \neq q$ (cancellation must be justified for M1, can be awarded if used correctly on $(x-q)^2(x-p-r+q)$ later) $\Leftrightarrow (x-q)(x-p-r+q) = 0$ $\Leftrightarrow x = p + r - q$ or $x = q$ Therefore there is only one point of intersection between the tangent and the curve if and only if $p + r - q = q$, which is if and only if the tangents are parallel. *One E mark for each direction.*	 B1 M1 A1 E1 M1 M1 M1 A1 E1 E1 (AG) (10 marks)

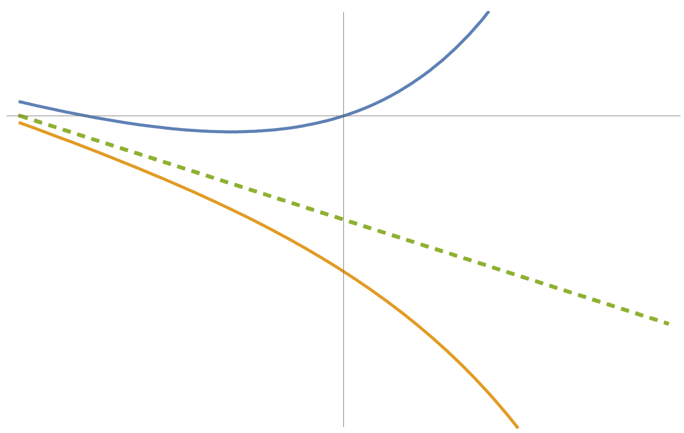
2			Sketch with areas $\int_0^x f(t)\,dt$, $\int_0^{f(x)} f^{-1}(y)\,dy$ and rectangle correctly identified. (One mark any one)	G1 G1 (2 marks)
	(i)		$g(0)(g(0)^2+1)=0$ factorised $g(0)$ real so $g(0)=0$ (must be justified) $1=(3g(t)^2+1)g'(t)$ $(3g(t)^2+1)>0$ so $g'(t)>0$ $g(2)^3+g(2)-2=0$ $(g(2)-1)(g(2)^2+g(2)+2)=0$ $\Delta=-7<0$ so $g(2)=1$ or g(2)>0 justified $g^{-1}(s)=s^3+s$ $\int_0^2 g(t)dt=2g(2)-\int_0^{g(2)} g^{-1}(s)ds$ $=\frac{5}{4}$	M1 A1 (AG) M1 A1 (AG) M1 A1 B1 M1 A1 (9 marks)
	(ii)		$h(t)=g(t+2)$ so $h(0)=g(2)=1$ and $h'(t)>0$ $(h(8)-2)(h(8)^2+2h(8)+5)=0$ $h(8)=2$ correctly justified $h^{-1}(s)=s^3+s-2$ $\int_0^8 h(t)dt+\int_{h(0)}^{h(8)} h^{-1}(s)ds=16$ (or similar correct equation) $\int_0^8 h(t)dt=16-\int_1^2(s^3+s-2)ds$ $=16-[\frac{s^4}{4}+\frac{s^2}{2}-2s]_1^2$ (integration) $=12\frac{3}{4}$	M1 A1 M1 A1 B1 M1 A1 M1 A1 (9 marks)

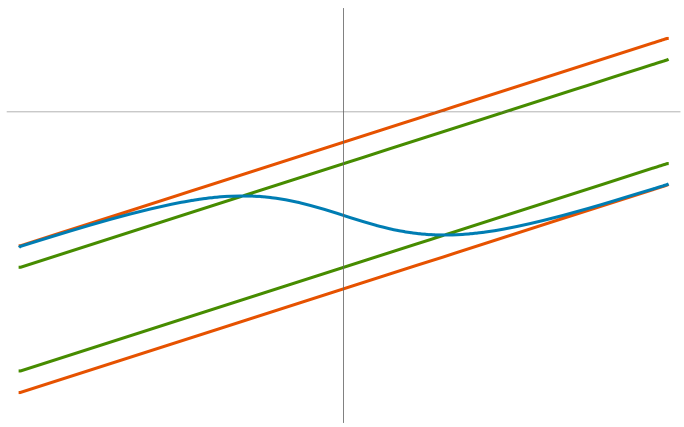
3			$\lvert x_1 + x_2\rvert$ is maximised when both have the same sign, In which case $\lvert x_1 + x_2\rvert = \lvert x_1\rvert + \lvert x_2\rvert$. Thus, $\lvert x_1 + x_2\rvert \le \lvert x_1 + x_2\rvert$ (or by consideration of all four combinations of signs separately) $\lvert x_1 + \cdots + x_{n-1} + x_n\rvert \le \lvert x_1 + \cdots + x_{n-1}\rvert + \lvert x_n\rvert$ $\le \cdots$ $\le \lvert x_1\rvert + \cdots + \lvert x_{n-1}\rvert + \lvert x_n\rvert$ by induction	E1 E1 (2 marks)
	(i)	(a)	$\lvert f(x) - 1\rvert = \lvert a_1x + \cdots + a_{n-1}x^{n-1} + x^n\rvert$ $\le \lvert a_1x\rvert + \cdots + \lvert a_{n-1}x^{n-1}\rvert + \lvert x^n\rvert$ $= \lvert a_1\rvert\lvert x\rvert + \cdots + \lvert a_{n-1}\rvert\lvert x\rvert^{n-1} + \lvert x\rvert^n$ $\le A(\lvert x\rvert + \cdots + \lvert x\rvert^{n-1}) + \lvert x\rvert^n$ $\le A(\lvert x\rvert + \cdots + \lvert x\rvert^{n-1} + \lvert x\rvert^n)$ (justified) $= A\frac{\lvert x\rvert(1-\lvert x\rvert^n)}{1-\lvert x\rvert}$ $\le A\frac{\lvert x\rvert}{1-\lvert x\rvert}$ (justified)	 M1 M1 M1 M1 M1 A1 (AG) (6 marks)
		(b)	$1 \le \frac{A\lvert\omega\rvert}{1-\lvert\omega\rvert}$ using $f(\omega) = 0$ $1 \le (A+1)\lvert\omega\rvert$ (with sign of $1 - \lvert\omega\rvert$ justified) $A + 1 \ge 1 \ge \lvert w\rvert$	M1 A1 (AG) B1 (AG) (3 marks)
		(c)	If $\lvert\omega\rvert > 1$, $0 = \omega^n f\left(\frac{1}{\omega}\right)$ $= 1 + a_{n-1}\omega + \cdots + a_1\omega^{n-1} + \omega^n$ Inequalities continue to hold since $\lvert a_i\rvert \le A$ If $\lvert\omega\rvert = 1$, then $1 + A \ge 1 \ge \frac{1}{1+A}$ since $A > 0$	 M1 E1 E1 (3 marks)
	(ii)		$f(x) = x^5 - x^4 - \frac{100}{135}x^3 - \frac{91}{135}x^2 - \frac{126}{135}x + 1$ Use $A = 1$. Integer roots with $\frac{1}{2} \le \lvert\omega\rvert \le 2$ could only be ±1 or ±2 $f(\pm2) \neq 0$ because numerator is odd (or any valid justification) $f(1) = -\frac{182}{135} \neq 0$ $f(1) = 0$ $x = 1$ is the only integer root.	B1 M1 M1 E1 A1 A1 (6 marks)

4	(i)		$\sin\frac{\pi}{9}\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$ $=\frac{1}{2}\sin\frac{2\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$ $=\frac{1}{8}\sin\frac{8\pi}{9}$ $=\frac{1}{8}\sin\frac{\pi}{9}$ (use of $\sin(\pi-x)=\sin(x)$) $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}=\frac{1}{8}$	B1 M1 M1 A1 (4 marks)
	(ii)		$\sin\left(\frac{x}{2^n}\right)\prod_{k=0}^{n}\cos\left(\frac{x}{2^k}\right)$ $=\frac{1}{2}\sin\left(\frac{x}{2^{n-1}}\right)\prod_{k=0}^{n-1}\cos\left(\frac{x}{2^k}\right)$ $=\cdots$ (convincing use of induction or repeated application) $=\frac{\sin(2x)}{2^{n+1}}$ (induction end point correct) $\prod_{k=0}^{n}\cos\left(\frac{x}{2^k}\right)=\frac{\sin(2x)}{2^{n+1}\sin\left(\frac{x}{2^n}\right)}$ $\sum_{k=0}^{n}\log\left(cos\left(\frac{x}{2^k}\right)\right)=\log(\sin(2x))-\log\left(sin\left(\frac{x}{2^n}\right)\right)-\log(2^{n+1})$ $\sum_{k=0}^{n}\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)=-2\cot(2x)+\frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)$ (justified with differentiation)	B1 M1 E1 A1 M1 (diff) M1 (division) A1 (AG) (7 marks)
	(iii)		B1 – switch to product starting at 0 M1 – set up as limiting case of product to n M1 – apply small angle for sin A1 – correct answer $\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)=\frac{\sin(2x)}{2^{n+1}\sin\left(\frac{x}{2^n}\right)\cos(x)}$ $=\frac{2\sin(x)}{2^{n+1}\sin\left(\frac{x}{2^n}\right)}$ $\sim\frac{\sin(x)}{2^n\times\left(\frac{x}{2^n}\right)}$ $=\frac{\sin(x)}{x}$	 M1 M1 M1 A1 (AG)

			$\sum_{j=2}^{n} \frac{1}{2^{j-2}} \tan\left(\frac{\pi}{2^j}\right)$ $= \sum_{k=0}^{n} \frac{1}{2^k} \tan\left(\frac{\pi/4}{2^k}\right)$ $= \lim_{n\to\infty} \left(\frac{1}{2^n} \cot\left(\frac{\pi/4}{2^n}\right) - 2\cot\left(\frac{\pi}{2}\right)\right)$ $= \lim_{n\to\infty} \left(\frac{1}{2^n \tan\left(\frac{\pi/4}{2^n}\right)}\right)$ $= \lim_{n\to\infty} \left(\frac{1}{2^n \left(\frac{\pi/4}{2^n}\right)}\right)$ $= \frac{4}{\pi}$	M1 M1 M1 M1 A1 (9 marks)

5	(i)		Constant iff $a = f(a)$ $\Leftrightarrow a = p + (a - p)a$ $\Leftrightarrow 0 = (a - p)(a - 1)$ $\Leftrightarrow a = p$ or $a = 1$. Period 2 $\Leftrightarrow a = f(f(a))$ $\Leftrightarrow 0 = (a - p)(-1 + 2ap - pa^2 + a^3)$ (factorisation) $\Leftrightarrow 0 = (a - p)(a - 1)(a^2 + (1 - p)a + 1)$ If $a = p$ or $a = 1$, then sequence is constant. The quadratic has solutions when $(p - 1)^2 \geq 4$. If $(p - 1)^2 > 4$, i.e. $p > 3$ or $p < -1$, the solutions are distinct. They are not both $1, p$ since the sum of the roots is $p - 1 \neq p + 1$ So for $p > 3$ or $p < -1$, one of the roots of the quadratic gives a sequence of period 2. If $p = 3$, $a = 1$ so not period 2. If $p = -1$, $a = -1 = p$ so not period 2.	M1 M1 A1 M1 M1 A1 B1 M1 E1 E1 (AG) B1 B1 (12 marks)
	(ii)		No value of a for which the sequence is constant $\Leftrightarrow f(a) = a$ has no solution $\Leftrightarrow f(x) > x$ or $f(x) < x$ for all x But $f(x) > x$ for large x. So cannot have $f(x) < x$ for all x. If no value of a for which sequence constant, then $f(x) > x$ for all x So $f\big(f(x)\big) > f(x) > x$ for all x And hence no solution to $f\big(f(a)\big) = a$. Setting $p = q$, gives (i). Then if $-1 \leq p \leq 3$, there is no period 2 sequence but a constant sequence exists.	 E1 (→) E1 (←) E1 E1 E1 E1 E1 E1 (8 marks)

6	(i)		If $y = mx + c$, Then the differential equation becomes $m = mx + c + x + 1$ $m = -1, c = -2$ $y = -x - 2$ $\frac{dy}{dx} = 0 \Rightarrow y + x + 1 = 0 \Rightarrow y = -x - 1$ $y = y_3(x)$ cannot cross the line $y = -x - 2$. So if $y_3(0) < -2$, it cannot reach the line $y = -x - 1$ and hence has no stationary points. At a stationary point, $\frac{d^2y}{dx^2} = \frac{dy}{dx} + 1 = y + x + 2 = 1 > 0$ so minimum $\frac{dY}{dx} = Y + 2$ $\log(Y + 2) = x + c$ $Y = -2 + Ae^x$ $y = -x - 2 + Ae^x$ $y(0) = 0 \Rightarrow A = 2$ $y(0) = -3 \Rightarrow A = -1$ (attempt at both) So $y = -x - 2 + 2e^x$ So $y = -x - 2 - e^x$ (both)  Curves tending to asymptote to the left Curve above line through origin tending to ∞ Curve below line tending to $-\infty$	 M1 A1 E1 (AG) E1 E1 M1 M1 A1 M1 G1 G1 G1 (12 marks)
	(ii)		If $y = mx + c$, Then the differential equation becomes $m = (mx + c)^2 + 4(mx + c) + x^2 - 4x - 2x(mx + c) + 3$ $0 = (m^2 - 2x + 1)x^2 + (2mc + 4m - 4 - 2c) + c^2 + 4c + 3 - m$ From x^2: $m = 1$ From x: $2mc + 4m - 4 - 2c = 2c + 4 - 4 - 2c = 0$	

			From 1: $c^2 + 4c + 2 = 0 \Rightarrow c = -2 \pm \sqrt{2}$ Any of these equations Correct values of m and c Solutions: $y = x - 2 \pm \sqrt{2}$ $\frac{dy}{dx} = (y-x)^2 + 4(y-x) + 3$ (writing as a function of $y - x$) $= (y - x + 3)(y - x + 1)$ Stationary pts: $y = x - 1$ or $y = x - 3$ Between these lines the gradient is negative. (Correctly justified) So stationary points on $y = x - 1$ are maxima and stationary on $y = x - 3$ are minima.  Curve does not intersect other solutions Curve has stationary points on correct lines	 M1 A1 M1 A1 A1 A1 G1 G1 (8 marks)

7	(i)		$\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) = 0$ $\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c} = -1$ <u>and cyclic permutations</u> $\boldsymbol{a} \cdot \boldsymbol{b} = -\frac{1}{2}$ legitimately obtained $\cos\theta = -\frac{1}{2}$ where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ $\theta = 120°$ Similarly, the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $120°$. Justification of equilateral triangle by sketch or otherwise ABC is equilateral	M1 M1 A1 M1 A1 M1 M1 A1 (8 marks)
	(ii)		$\boldsymbol{a_1} \cdot \boldsymbol{a_2} + \boldsymbol{a_1} \cdot \boldsymbol{a_3} + \boldsymbol{a_1} \cdot \boldsymbol{a_4} = -1$ <u>and cyclic permutations</u> Linear combination of these equations $\boldsymbol{a_1} \cdot \boldsymbol{a_2} = \boldsymbol{a_3} \cdot \boldsymbol{a_4}$ (legitimately obtained)	M1 M1 A1 (AG) (3 marks)
		(a)	Angles $\angle A_1OA_2 = \angle A_3OA_4$ By symmetry, $\angle A_2OA_3 = \angle A_4OA_1$ The $\boldsymbol{a_i}$ are distinct and unit length so no angles are zero (accept justification by sketch) $A_1A_2A_3A_4$ is a rectangle	M1 M1 A1 (3 marks)
		(b)	$(A_1A_2)^2 = (\boldsymbol{a_1} - \boldsymbol{a_2})^2$ $= \boldsymbol{a_1^2} + \boldsymbol{a_1^2} - 2\boldsymbol{a_1} \cdot \boldsymbol{a_2}$ $= 2 - 2\boldsymbol{a_1} \cdot \boldsymbol{a_2}$ By symmetry, $\boldsymbol{a_1} \cdot \boldsymbol{a_2} = \boldsymbol{a_1} \cdot \boldsymbol{a_3} = \boldsymbol{a_1} \cdot \boldsymbol{a_4}$ So $\boldsymbol{a_1} \cdot \boldsymbol{a_2} = -\frac{1}{3}$ So $(A_1A_2)^2 = \frac{8}{3}$ $A_1A_2 = \frac{2\sqrt{2}}{\sqrt{3}}$	 M1 M1 M1 A1 M1 A1 (6 marks)

8	(i)		$f(\boldsymbol{M}) = f(\boldsymbol{MI}) = f(\boldsymbol{M})f(\boldsymbol{I})$ $\Rightarrow f(\boldsymbol{I}) = 1$ since $f(\boldsymbol{M}) \neq 0$	M1 A1 (AG) (2 marks)
	(ii)		$f(J)^2 = f(J^2)$ $= f(I) = 1$ $\Rightarrow f(J) = -1$ since $f(J) \neq 1$ $f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) = f\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ $= -f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ legitimately obtained $f\left(\begin{pmatrix} d & c \\ b & a \end{pmatrix}\right) = f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ $= -f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right)$ legitimately obtained	M1 M1 A1 M1 A1 (AG) M1 A1 (AG) (7 marks)
	(iii)		Using first equality in previous part (or otherwise correctly justified) $f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = -f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right)$ $f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = 0$ $f\left(\begin{pmatrix} a & b \\ ka & kb \end{pmatrix}\right) = f\left(\boldsymbol{K}\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right)$ $= 0$	M1 M1 M1 A1 (AG) (4 marks)
	(iv)		$\boldsymbol{K}^{-1}\boldsymbol{PK} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ $f(\boldsymbol{K})f(\boldsymbol{K}^{-1}) = F(\boldsymbol{I}) = 1 \Rightarrow f(\boldsymbol{K}^{-1}) = f(\boldsymbol{K})^{-1}$ $f(\boldsymbol{K}^{-1}\boldsymbol{PK}) = f(\boldsymbol{K}^{-1})f(\boldsymbol{PK})$ (must use two stages) $= f(\boldsymbol{K}^{-1})f(\boldsymbol{P})f(\boldsymbol{K})$ $= f(\boldsymbol{P})$ $\boldsymbol{P}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $f(\boldsymbol{P}^2) = f(\boldsymbol{P}) \Rightarrow f(\boldsymbol{P}) = 0$ or 1 $\boldsymbol{P}^{-1}$ exists so $f(\boldsymbol{P})f(\boldsymbol{P}^{-1}) = 1 \Rightarrow f(\boldsymbol{P}) \neq 0$	B1 M1 M1 A1 (AG)? B1 M1 A1 (7 marks)

9	(i)		$\boldsymbol{r} = \begin{pmatrix} ut\cos\alpha \\ ut\sin\alpha - \frac{1}{2}gt^2 \end{pmatrix}$	M1
			$\boldsymbol{r}^2 = u^2t^2\cos^2\alpha + u^2t^2\sin^2\alpha - ugt^3\sin\alpha + \frac{1}{4}g^2t^4$	M1
			$= u^2t^2 - ugt^3\sin\alpha + \frac{1}{4}g^2t^4$	A1
			$\frac{d}{dt}(\boldsymbol{r}^2) = 2u^2t - 3ugt^2\sin\alpha + g^2t^3$	M1 A1
			$= t(2u^2 - 3ugt\sin\alpha + g^2t^2)$	M1
			$= t(2u^2 - \frac{9}{4}u^2\sin^2\alpha + (gt - \frac{3}{2}u\sin\alpha)^2)$	M1
			If $\sin\alpha < \frac{2\sqrt{2}}{3}$, then $2u^2 - \frac{9}{4}u^2\sin^2\alpha > 0$	M1
			and distance is always increasing.	A1 (AG)
			If $\sin\alpha > \frac{2\sqrt{2}}{3}$, then distance is decreasing at $t = \frac{3u}{2g}\sin\alpha$	M1
			Landing occurs at $t = \frac{2u}{g}\sin\alpha$, which is later	E1
			(<u>Or</u> imagine falls through ground. Distance increasing while underground, so any decrease must be above ground)	(11 marks)
	(ii)		$\boldsymbol{r} = \begin{pmatrix} ut\cos\alpha + vt \\ ut\sin\alpha - \frac{1}{2}gt^2 \end{pmatrix}$	B1
			$PQ^2 = (u\cos\alpha + v)^2t^2 + u^2t^2\sin^2\alpha - ugt^3\sin\alpha + \frac{1}{4}g^2t^4$	M1 A1
			$\frac{d}{dt}(PQ^2) = 2t(u\cos\alpha + v)^2 + 2u^2t\sin^2\alpha + 2tu^2\sin^2\alpha - 3ugt^2\sin\alpha + g^2t^3$	M1
			$= t\left(2(u\cos\alpha + v)^2 + 2u^2\sin^2\alpha - \frac{9}{4}u^2\sin^2\alpha + \left(gt - \frac{3}{2}u\sin\alpha\right)^2\right)$	M1
			$= t\left(2(u\cos\alpha + v)^2 - \frac{1}{4}u^2\sin^2\alpha + \left(gt - \frac{3}{2}u\sin\alpha\right)^2\right)$	M1 A1
			If $2\sqrt{2}v > (\sin\alpha - 2\sqrt{2}\cos\alpha)u$, then $8(u\cos\alpha + v)^2 > u^2\sin^2\alpha$	M1
			So PQ is increasing for all t.	A1 (AG)
				(9 marks)

10	(i)		Correct diagram Moments about A: $Wa\cos\theta\,(1+2k) = 2aT\sin 2\theta$ If $2k+1 > 4\sin\theta$ then $2T\sin 2\theta > W\cos\theta\,(4\sin\theta) = 2W\sin 2\theta$ Since $\sin 2\theta > 0$, $T > W$ and so the string will break.	B2 M1 M1 A1 A1 (AG) (6 marks)
	(ii)		Resolving forces vertically: $R = ((k+1)W - T\sin\theta)$ Resolving horizontally, ring will slip if: $T\cos\theta > \mu\big((k+1)W - T\sin\theta\big)$ (= max value for friction) Moments about A: $W(2k+1) = 4T\sin\theta$ $\mu\big((k+1)W - T\sin\theta\big) = \mu\left(\frac{4(k+1)}{2k+1} - 1\right)T\sin\theta$ $\mu\left(\frac{2k+3}{2k+1}\right)T\sin\theta$ If $2k+1 > (2k+3)\mu\tan\theta$, then $\mu\left(\frac{2k+3}{2k+1}\right)\sin\theta < \cos\theta$ So the ring will slip.	 M1 M1 M1 A1 M1 A1 (AG) (6 marks)
	(iii)		Attempt to solve breaking inequality for k Breaks at $k = \frac{4\sin\theta - 1}{2}$ Attempt to solve slipping inequality for k Slips at $k = \frac{3\mu\tan\theta - 1}{2(1-\mu\tan\theta)}$ If ring slips before it breaks: $\frac{3\mu\tan\theta - 1}{2(1-\mu\tan\theta)} < \frac{4\sin\theta - 1}{2}$ (for A1, do not allow >) Confirming that inequality is being multiplied by a positive quantity. $3\mu\tan\theta - 1 < 4\sin\theta - 1$ $\mu < \frac{2\cos\theta}{2\sin\theta + 1}$	M1 A1 B1 M1 A1 E1 M1 A1 (AG) (8 marks)

11	(i)		In both cases, award the M mark if all possible values of n_2 for at least 3 values of n_1 are identified. $\boldsymbol{n_3 = 9}$ $n_1 = 1; n_2$ has no options $n_1 = 2; n_2 = 8$ $n_1 = 3; n_2 = 8, 7$ $n_1 = 4; n_2 = 8, 7, 6$ $n_1 = 5; n_2 = 8, 7, 6$ $n_1 = 6; n_2 = 8, 7$ $n_1 = 7; n_2 = 8$ $n_1 = 8; n_2$ has no options $\text{Total} = (1 + 2 + 3) \times 2 = 12$ $\boldsymbol{n_3 = 10}$ $n_1 = 1; n_2$ has no options $n_1 = 2; n_2 = 9$ $n_1 = 3; n_2 = 9, 8$ $n_1 = 4; n_2 = 9, 8, 7$ $n_1 = 5; n_2 = 9, 8, 7, 6$ $n_1 = 6; n_2 = 9, 8, 7$ $n_1 = 7; n_2 = 9, 8$ $n_1 = 8; n_2 = 9$ $n_1 = 0; n_2$ has no options $\text{Total} = (1 + 2 + 3 + 4) \times 2 - 4 = 16$ $\boldsymbol{n_3 = 2n + 1}$ Total ways $= \big(1 + \cdots + (n-1)\big) \times 2$ (method mark may be implicit) $= (n-1)n$ $\boldsymbol{n_3 = 2n}$ Total ways $= \big(1 + \cdots + (n-1)\big) \times 2 - (n-1)$ (method mark may be implicit) $= (n-1)^2$	M1 M1 A1 (both totals correct) M1 A1 M1 A1 (7 marks)
	(ii)		Total number of pairs is $\binom{N-1}{2} = \frac{1}{2}(N-1)(N-2)$ Justification for using first part of question $\boldsymbol{N = 2n + 1}$ $\text{Prob} = \frac{(n-1)n}{\frac{1}{2}(2n)(2n-1)} = \frac{n-1}{2n-1}$ $\boldsymbol{N = 2n}$ $\text{Prob} = \frac{(n-1)^2}{\frac{1}{2}(2n-1)(2n-2)} = \frac{n-1}{2n-1}$	M1 B1 A1 (AG) A1 (4 marks)

	(iii)		$\text{Prob} = \sum_{n=1}^{M} \frac{n-1}{2n-1} \times \mathbb{P}(\text{largest rod is } 2n+1) + \sum_{n=1}^{M} \frac{n-1}{2n-1} \times \mathbb{P}(\text{largest rod is } 2n)$	M1 A1 (ft)
			$= \sum_{n=1}^{M} \frac{n-1}{2n-1}\left(\frac{\binom{2n}{2}}{\binom{2M+1}{3}} + \frac{\binom{2n-1}{2}}{\binom{2M+1}{3}}\right)$	
			$= \frac{6}{(2M+1)(2M)(2M-1)} \cdot \frac{1}{2}\sum_{n=1}^{M} \frac{n-1}{2n-1}(2n(2n-1) + (2n-1)(2n-2))$ (Use of formula for binomial coefficients with factorials cancelled)	M1 A1
			$= \frac{3}{M(2M+1)(2M-1)}\sum_{n=1}^{M}(n-1)\,(2n-1)$	M1
			Use of $\sum_1^K k^2 = \frac{1}{6}K(K+1)(2K+1)$ to simplify above	M1
			$= \frac{3}{M(2M+1)(2M-1)}\left(\frac{1}{3}M(M+1)(2M+1) - 3 \times \frac{1}{2}M(M+1) + M\right)$ $= \frac{1}{2(2M+1)(2M-1)}(4M^2 - 3M - 1)$ $= \frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}$	M1
				M1 A1 (9 marks)

12	(i)		$\mu = \int_0^1 nx^n \, dx = \frac{n}{n+1}$ $\mathbb{E}(X^2) = \int_0^1 nx^{n+1} \, dx = \frac{n}{n+2}$ $\sigma^2 = \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2 = \frac{n}{(n+1)^2(n+2)}$	M1 A1 M1 M1 A1 (AG) (5 marks)
	(ii)		$\text{LQ} = \frac{1}{2}, \text{UQ} = \frac{\sqrt{3}}{2}$ $\text{IQR} = \frac{\sqrt{3}-1}{2}$ $2\sigma = \frac{\sqrt{2}}{3}$ Squaring IQR and 2σ Comparing $\sqrt{3}$ with a rational number… …by squaring both sides Argument correct	M1 A1 B1 M1 M1 M1 A1 (7 marks)
	(iii)		$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$ $+ \frac{n(n-1)\cdots(n-k+1)}{k!}x^k + \cdots$ $\text{LQ} = \left(\frac{1}{4}\right)^{1/n}$ and $\text{Median} = \left(\frac{1}{2}\right)^{1/n}$ $\left(\frac{1}{\mu}\right)^n = \left(1+\frac{1}{n}\right)^n > 1 + n\left(\frac{1}{n}\right) = 2$ So $\mu < \left(\frac{1}{2}\right)^{1/n}$ $\left(\frac{1}{\mu}\right)^n = \left(1+\frac{1}{n}\right)^n < 1 + n\left(\frac{1}{n}\right) + \frac{n^2}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \frac{n^k}{k!}\left(\frac{1}{n}\right)^k + \cdots$ $< 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!} + \cdots$ < 4 So $\mu > \left(\frac{1}{4}\right)^{1/n}$	A1 A1 B1 M1 A1 M1 M1 A1 (8 marks)

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